Using quadratics to find the range of some rational functions

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Example

Find the range of the function $f(x) = \dfrac{x+6}{x^2+x-6}$.

Solution Let $y = \dfrac{x+6}{x^2+x-6}$

$$y(x^2+x-6) = x+6$$
$$x^2y + xy - 6y = x+6$$
$$yx^2 + (y-1)x - (6y+6) = 0$$

Since x is real, $\therefore \ \Delta \ge 0$.

$$\therefore \ (y-1)^2 - 4y[-(6y+6)] \ge 0$$

$$y^2 - 2y + 1 + 24y^2 + 24y \ge 0$$
$$25y^2 + 22y + 1 \ge 0 \quad \ldots. \quad (1)$$

For the equation, $25y^2 + 22y + 1 = 0$, the roots are $y = \dfrac{-11 \pm 4\sqrt{6}}{25}$

The inequality (1) becomes $25\left(y - \dfrac{-11-4\sqrt{6}}{25}\right)\left(y - \dfrac{-11+4\sqrt{6}}{25}\right) \ge 0$

$$y \le \frac{-11-4\sqrt{6}}{25} \approx -0.83192 \quad or \quad y \ge \frac{-11+4\sqrt{6}}{25} \approx -0.048082$$

Hence the range of *f*(x) is $\left(-\infty, \dfrac{-11-4\sqrt{6}}{25}\right]$ *or* $\left[\dfrac{-11+4\sqrt{6}}{25}, +\infty\right)$

Graph for reference :